# A nonstabilizerness monotone from stabilizerness asymmetry

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MBQM 2024

NOVEMBER 18, 2024

PST, M. Frau, T. Haug, E. Tirrito, and L. Piroli, arxiv:2411.05766

# Outline

- Preliminaries: stabilizer group, stabilizer basis, magic monotones
- Asymmetry
- Basis-minimized stabilizerness asymmetry (BMSA): properties, connection with known quantities, numerical methods

# Stabilizer group

- Given a pure $N$-qubit stabilizer state $|s\rangle$, the unsigned stabilizer group is

$$G(|s\rangle) = \{P \in \mathcal{P}_N : P|s\rangle = \pm|s\rangle\}$$

- $\mathcal{P}_N$ : the set of Pauli strings with trivial phase $+1$
- $G(|s\rangle)$ is generated by $N$ mutually commuting Pauli strings

# Stabilizer basis

- Given a stabilizer group $G$, the set of stabilizer states $\{|s\rangle : G(|s\rangle) = G\}$ forms an orthornormal basis, called stabilizer basis
- This basis can be constructed by introducing destabilizers $d_i$, such that the stabilizer states above can be written as $\{d_i|s\rangle\}_i$
- For any state, we can write

$$|\psi\rangle = \sum_i c_i d_i |s\rangle$$

Aaronson, Gottesman, PRA (2024)

# Magic monotones

A magic monotone $M(|\psi\rangle)$ has to satisfy the following properties:

Faithful: $M(|\psi\rangle) = 0$ iff $|\psi\rangle$ is a STAB

Sub-additive: $M(|\psi\rangle \otimes |\phi\rangle) \leq M(|\psi\rangle) + M(|\phi\rangle)$

Non-increasing under Clifford operations:

$$M(\Gamma|\psi\rangle) \leq M(|\psi\rangle), \Gamma \in C$$

Non-increasing on average under measurements (strong monotonicity):

$$\sum_i p_i\, M(K_i \rho K_i^*) \leq M(|\psi\rangle)$$

# Why strong monotones?

- Question: can we transform $m$ copies of a resource state $\rho$ to $k$ copies of a target state $\sigma$ using Clifford operations?
- With deterministic transformation, then we must have

$$M(\rho^m) \geq M\left(\sigma^k\right)$$

- With stochastic transformation with success probability $p$, if $M$ is a strong monotone,

$$M(\rho^m) \geq pM\left(\sigma^k\right)$$

- In the asymptotic limit (infinitely many copies), we define the regularized version

$$M^\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} M(\rho^n)$$

Veitch *et al*, NJP **16** 013009 (2014)

# Known magic monotones

- Most known magic monotones require optimization procedures to compute, thus difficult to compute beyond a few qubits
- Strong but not computable: relative entropy of magic, robustness of magic
- Computable but not strong: stabilizer Renyi entropy (SRE)
- Can we construct a strong and computable monotone?

# Stabilizer Renyi entropy (SRE)

- A magic monotone for pure states based on expectation values of Pauli strings
- For a pure state $|\psi\rangle$, SRE is defined as

$$M_n(|\psi\rangle) = (1-n)^{-1} \log \sum_{P \in \mathcal{P}_N} \Xi_P^n(|\psi\rangle) - N \log 2\,,$$

where $\Xi_P(|\psi\rangle) = \langle\psi|P|\psi\rangle^2 / 2^N$.

- A good monotone for integer $n \geq 2$

Leone, Oliviero, Hamma, PRL (2020)

Leone, Oliviero, Hamma, PRL **128**, 00402 (2022)

# Violation of strong monotonicity

- A counter example of strong monotonicity for SRE of any index can be shown using the state

$$|\psi^{\varepsilon}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\varepsilon}}}\left[|0\rangle^{\otimes N} + \varepsilon\,|\chi\rangle^{\otimes N}\right] ,$$

where $|\chi\rangle = e^{-i(\pi/4)}\cos\beta|0\rangle + \sin\beta|1\rangle\,,$

- After projective measurement on the first qubit, there is a finite probability that the state is projected to

$$|\psi'_N\rangle = |1\rangle \otimes |\chi\rangle^{\otimes(N-1)}$$

- The SRE is $O(1)$, which becomes $O(N)$ on average

Haug, Piroli, Quantum 7, 1092 (2023)

# Stabilizer linear entropy

- Stabilizer linear entropy defined as

$$M_n^{\text{lin}} = 1 - \exp(-M_n)$$

  was shown to be a strong monotone

- However, it is not additive: it can only take values between 0 and 1

- The regularized version of $M_n^{\text{lin}}$ is always zero → not useful for asymptotic state transformation

Leone, Bittel, PRA 110 L040403 (2024)

Leone, Bittel, arxiv:2404.11652

# Strong monotones

Generic monotones from resource theory:

- Relative entropy of magic

Veitch *et al*, NJP **16** 013009 (2014)

$$\mathrm{r}_{\mathcal{M}}(\rho) \equiv \min_{\sigma \in \mathrm{STAB}(\mathcal{H}_{\mathrm{d}})} S(\rho \| \sigma)$$

- Robustness of magic

Howard and Campbell, PRL **118**, 090501 (2017)

$$\mathcal{R}(\rho) := \min_{\sigma_+, \sigma_- \in \mathrm{STAB}_n} \left\{ 2p+1 \,\middle|\, \rho = (p+1)\sigma_+ - p\sigma_-, \ p \geq 0 \right\}.$$

# Asymmetry

- Given a group $G$, the $G$-asymmetry measures the asymmetry of a state $\rho$ with respect to $G$, i.e., how much it deviates from the $G$-symmetric states
- In recent years, the quantity has been used to probe symmetry breaking by the name of "entanglement asymmetry"
- Formally, $G$-asymmetry is a strong monotone in the resource theory of $G$-frameness

Gour et al, PRA **80**, 012307 (2009)

# Asymmetry

- For a discrete group $G$, we consider the symmetrized state (via $G$-twirling operation)

$$\mathcal{G}_G(\rho) = \frac{1}{|G|} \sum_{g \in G} U_g \rho U_g^\dagger,$$

- The $G$-asymmetry is defined as

$$\begin{aligned} A_G(\rho) &= S(\rho \| \mathcal{G}_G(\rho)) \\ &= S(\mathcal{G}_G(\rho)) - S(\rho). \end{aligned}$$

$S(\rho\|\sigma)$ : quantum relative entropy

- It has the following properties
    1. It is non-negative, $A_G(\rho) \geq 0$
    2. It vanishes if and only if $[\rho, G] = 0$

Gour et al, PRA **80**, 012307 (2009)

# Asymmetry

- Moreover, it is a strong monotone in the resource theory of $G$-frameness

$$A_{G(|S\rangle)}(\rho) \geq \sum_{i\in\{+,-\}} p_i A_{G(|S\rangle)}(\rho_i).$$

- The Renyi version can be defined as

$$A_{G,\alpha}(\rho) = S_\alpha(\mathcal{G}_G(\rho)) - S_\alpha(\rho),$$

which also has the two properties above, but generally not strong monotonicity.

Gour et al, PRA **80**, 012307 (2009)

# Asymmetry for groups $G \subset P_N$

- If the group $G$ is generated by Pauli operators $P_1, \dots, P_k$, then

$$\mathcal{G}_G(\rho) = \frac{1}{2^N} \sum_{P \in G^\perp} \mathrm{Tr}[\rho P] P.$$

where $G^\perp$ is the group of Pauli operators that commute with $G$.

- For $\alpha = 2$, it takes a nice form

$$A_{G,2}(\rho) = -\log_2 \frac{\sum_{P \in G^\perp} |\mathrm{Tr}[\rho P]|^2}{\sum_{P \in \mathcal{P}_N} |\mathrm{Tr}[\rho P]|^2}.$$

it can be interpreted as the probability that a Pauli string sampled as $\mathrm{Tr}\,(\rho P)^2/2^N$ commutes with all Pauli strings in $G$

# Stabilizerness asymmetry

If $G(|s\rangle)$ is the unsigned stabilizer group of a stabilizer state $|s\rangle$:

$$G(|s\rangle) = \{P \in \mathcal{P}_N : P|s\rangle = \pm|s\rangle\}$$

$$\begin{aligned}\mathcal{G}_{G(|s\rangle)}(\rho) &= \frac{1}{2^N} \sum_{P \in G(|s\rangle)} P\rho P^{-1} \\ &= \frac{1}{2^N} \sum_{P \in G(|s\rangle)} \mathrm{Tr}[\rho P] P,\end{aligned}$$

then the $G$-asymmetry vanishes if and only if $|\psi\rangle$ is stabilized by $G(|s\rangle)$

# Basis minimized stabilizerness asymmetry (BMSA)

- For pure states, we define the BMSA as

$$\mathcal{A}_\alpha(|\psi\rangle) = \min_{|S\rangle \in \mathrm{PSTAB}_N} A_{G(|S\rangle),\alpha}(\rho)$$
$$= \min_{|S\rangle \in \mathrm{PSTAB}_N} S_\alpha(\mathcal{G}_{G(|S\rangle)}(\rho)).$$

- It has the following properties
  1. It vanishes if and only if $|\psi\rangle$ is a stabilizer state
  2. It is invariant under Clifford unitaries
  3. It is sub-additive
  4. It does not increase on average under Pauli measurement (for $\alpha = 1$)
  5. It does not increase under partial tracing (for $\alpha = 1$)

# Convex-roof extension

We can extend the BMSA to mixed states based on convex-roof:

$$\mathcal{A}(\rho) = \min_{\{p_i,\rho_i\}} \sum_i p_i \mathcal{A}(\rho_i),$$

where the minimum is taken over all possible convex decompositions of $\rho$

# Connection to basis-minimized measurement entropy

- Basis-minimized measurement entropy is defined as

$$z_q^*(|\psi\rangle) = \min_{C \in \mathcal{C}_N} S_q^{\text{part}}(C^\dagger |\psi\rangle),$$

where

$$S_q^{\text{part}}(|\psi\rangle) = \frac{1}{1-q} \log_2 \sum_\sigma |\langle \sigma | \psi \rangle|^{2q},$$

are the participation entropies.

- It can be shown that

$$\mathcal{A}_\alpha(|\psi\rangle) = z_\alpha^*(|\psi\rangle).$$

Niroula et al , Nat. Phys. (2024)

# Relation with SRE

Lower bound:

$$M_n(|\psi\rangle) \leq \frac{n}{n-1}\mathcal{A}_\alpha(|\psi\rangle) \quad (n > 1, \alpha \leq 2).$$

$$2\mathcal{A}_\alpha(|\psi\rangle) \geq M_n(|\psi\rangle) \quad (n \geq 1/2, \alpha \leq 1/2)$$

Upper bound:

There exists a constant $C > 1$ such that

$$\mathcal{A}_\alpha(|\psi\rangle) \leq 2CM_n(|\psi\rangle) \quad (n \leq 2, \alpha \geq 2)$$

# Connection to coherence

- The set of stabilizer states STAB is defined as

$$\mathrm{STAB} = \left\{ \rho : \rho = \sum_j p_j \left| S_j \right\rangle \left\langle S_j \right| , \forall j p_j \geq 0, \sum_j p_j = 1 \right\}$$

where $|S_j\rangle$ are pure stabilizer states.

# Connection to coherence

- Equivalently,

$$\mathrm{STAB} = \left\{ \rho : \rho = \sum_j p_j \tau_j, \forall j p_j \geq 0, \sum_j p_j = 1 \right\},$$

where $\tau_j$ are incoherent states in the stabilizer basis.

→ We can construct a measure of magic by minimizing a measure of coherence over all possible stabilizer basis

# Connection to coherence

BMSA:

- Relative entropy of coherence → $\alpha = 1$
- $l_1$ norm of coherence → $\alpha = \frac{1}{2}$

Both of them are strong coherence measures, thus leading to strong magic measures

# BMSA and classical simulations

- We would like to compute

$$\langle O\rangle = \langle 0^{\otimes N}|C_1^\dagger U_1^\dagger \cdots C_D^\dagger U_D^\dagger O U_D C_D \cdots U_1 C_1|0^{\otimes N}\rangle$$

$C_i$: Clifford gates, $U_j$ : non-Clifford Pauli rotation $\exp(i\theta P)$, $O$: Pauli operator

- The circuit above can be reduced to circuits containing only Pauli rotations $U_j$ through circuit compilation

# BMSA and classical simulations

- The action of $U_P(\theta) = \exp(i\theta P) = \cos(\theta)\ + i\sin(\theta)\,P$ to a stabilizer state is

$$U_P(\theta)\,|s\rangle = \begin{cases} \exp(i\theta\lambda_s)\,|s\rangle\,, & P \in G(|s\rangle) \\ \cos(\theta)\,|s\rangle + i\sin(\theta)\,|s'\rangle\,, & \text{otherwise} \end{cases}$$

→ each application of $U_P(\theta)$ may increase the number of coefficients by a factor of 2

- We can simulate the circuit by keeping track of the coefficients in a given stabilizer basis (and possibly minimized over)

- The simulation can be done approximately by truncating all coefficients below a given error threshold $|c_i| < \epsilon$

# BMSA and classical simulations

- The BMSA is related to the accuracy of such truncation: if $\mathcal{A}_1(|\psi\rangle)$ scales as $O(N)$, then for a given $\epsilon$, the required number of coefficients that needs to be retained scales as $O(\exp N)$
- The simulation technique can be seen as the Schrodinger picture version of the sparse Pauli dynamics method

T. Begusic, Sci Adv (2024)

# Methods to compute BMSA

1. Exact brute-force from Pauli vector (5 qubits)
2. Exact via branch and bound (9 qubits)
3. Estimation via minimization of participation entropy

# Estimation via minimization of participation entropy

- Motivated by CAMPS algorithm

X. Qian et al, arxiv (2024)

$|\tilde{\psi}\rangle = U_C|\psi\rangle$ → find Clifford gate $U_C$ that minimizes $S^{\text{part}}$

- Sweep over two-qubit Clifford unitaries on neighboring sites
- Need to consider 15 gates in $\mathcal{C}_2/\mathcal{C}_z$

$\mathcal{C}_z$: subgroup of Clifford group that maps Pauli-Z to Pauli-Z

# Estimation via minimization of participation entropy

- The probabilities for $C_i \in \mathcal{C}_2/\mathcal{C}_z$ are

$$p_{C_i} = \mathrm{e}^{-(S_\alpha^{\mathrm{part}}(C_i C_{\mathrm{ref}}|\psi\rangle) - S_\alpha^{\mathrm{part}}(C_{\mathrm{ref}}|\psi\rangle))/T}$$

where $T$ is fictitious temperature

- The temperature $T$ is slowly decreased to $T \to 0$
- Equivalent to simulated annealing

# Estimation via minimization of participation entropy

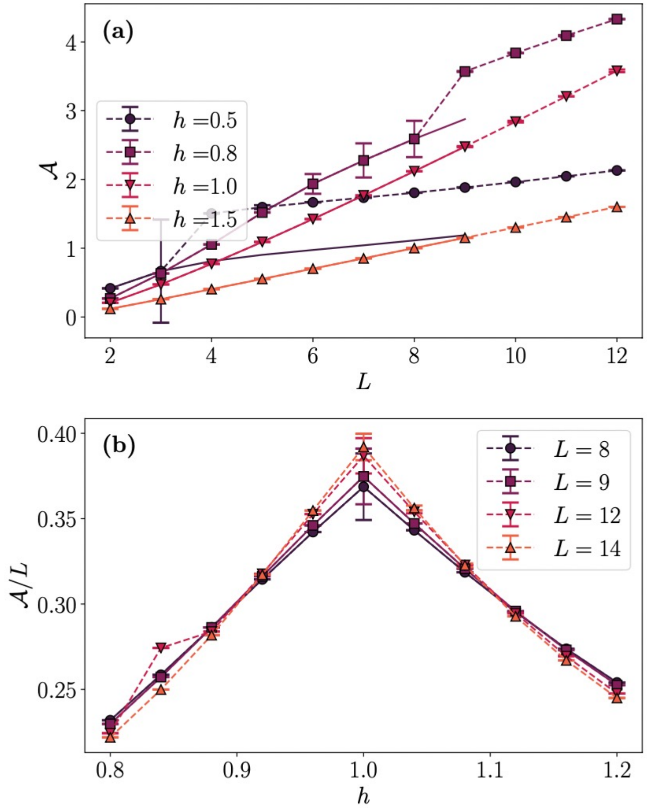
$$H_{\text{Ising}} = -\sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x ,$$

- In the ferro phase, better results are obtained if we initialize the Clifford circuit by

$$C_{\text{ref}} = H_1 CNOT_{1,2} CNOT_{2,3} \ldots CNOT_{N-1,N}$$

- Perfect agreement with exact values for $L =$ 8,9

- Able to reach $L =$ 12,14 accurately



# BMSA of W-state

$$|W\rangle = \frac{1}{\sqrt{N}}(|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle).$$

Upper bound: participation entropy

Lower bound: SRE-2

$$\frac{3}{2}\ln(N) - \frac{1}{2}\ln(7N-6) \leq \mathcal{A}_\alpha(|W\rangle) \leq \ln(N),$$

$$\mathcal{A}_\alpha(|W\rangle) \sim \ln(N)$$

# Conclusions

- We have introduced a measure of nonstabilizerness, the BMSA, which is a strong monotone and additive at large $N$
- The BMSA can be computed exactly up to $N = 9$ qubits, and we developed a Monte Carlo approach to estimate it for larger sizes
- We derived several inequalities between the BMSA and other known monotones, showing how these inequalities may allow one to infer the leading behavior of the BMSA in the large $N$ limit

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PST, M. Frau, T. Haug, E. Tirrito, and L. Piroli,
A nonstabilizerness monotone from stabilizerness asymmetry
arxiv:2411.05766

# THANK YOU